ABSOLUTE CONTINUITY OF THE INTEGRATED DENSITY OF STATES FOR THE ALMOST MATHIEU OPERATOR WITH NON-CRITICAL COUPLING

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Abstract. We show that the integrated density of states of the almost Mathieu operator is absolutely continuous if and only if the coupling is non-critical. We deduce for subcritical coupling that the spectrum is purely absolutely continuous for almost every phase, settling the measure-theoretical case of Problem 6 of Barry Simon's list of Schrödinger operator problems for the twenty-first century.

1. Introduction

This work is concerned with the almost Mathieu operator $H = H_{\lambda,\alpha,\theta}$ defined on $\ell^2(\mathbb{Z})$

$$(Hu)_n = u_{n+1} + u_{n-1} + 2\lambda \cos(2\pi[\theta + n\alpha])u_n \tag{1}$$

where $\lambda \neq 0$ is the coupling, $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ is the frequency, and $\theta \in \mathbb{R}$ is the phase. This is the most heavily studied quasiperiodic Schrödinger operator, arising naturally as a physical model (see [22] for a recent historical account and for the physics background).

We are interested in the integrated density of states, which can be defined as the limiting distribution of eigenvalues of the restriction of $H = H_{\lambda,\alpha,\theta}$ to large finite intervals. This limiting distribution (which exists by [7]) turns out to be a θ independent continuous increasing surjective function $N = N_{\lambda,\alpha} : \mathbb{R} \to [0, 1]$. The support of the probability measure dN is precisely the spectrum $\Sigma = \Sigma_{\lambda,\alpha}$. Another important quantity, the Lyapunov exponent $L = L_{\lambda,\alpha}$, is connected to the integrated density of states by the Thouless formula $L(E) = \int \ln |E - E'|\, dN(E')$.

Since Σ is a Cantor set ([8], [10], [21], [23], [3], [2]), N is "Devil's staircase"-like. It is also known that Σ has Lebesgue measure $|4 - 4|\lambda||$ [3]. See [22] for the history of these two problems and additional references. The spectrum has zero Lebesgue measure precisely when $|\lambda| = 1$, the critical coupling.

Recently there was quite a bit of interest in the regularity properties of the integrated density of states. The modulus of continuity is easily seen to be quite poor for generic frequencies, so positive results (such as Hölder continuity [13]) in this direction have depended on suitable (full measure) conditions on the frequency. There was more hope in proving positive general results on absolute continuity of

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N since [20] established that for almost every α and λ non-critical (or every α and certain values of λ), dN has an absolutely continuous component (for critical coupling, dN never has an absolutely continuous component due to zero Lebesgue measure of Σ). Later, this result was improved to full absolute continuity under a full measure condition on α and all non-critical couplings [17] (the condition on α was later improved in [2]). Let us also mention the recent work of Goldstein-Schlag [14] that establishes absolute continuity of the integrated density of states in the regime of positive Lyapunov exponent under a full measure condition on α, but for more general potentials.

In this work, we prove the following result, which completely describes the set of parameter values for which the integrated density of states is absolutely continuous.

Main Theorem. *The integrated density of states of $H_{\lambda,\alpha,\theta}$ is absolutely continuous if and only if $|\lambda| \neq 1$.*

We point out that prior to this result, it was unknown whether there could be parameters for which the spectrum contains pieces (non-empty open subsets) of both positive and zero Lebesgue measure.

Our Main Theorem has an important consequence regarding the nature of the spectral measures of the operators $H_{\lambda,\alpha,\theta}$. For $|\lambda| \geq 1$, it is known that the spectral measures have no absolutely continuous component. For $|\lambda| < 1$, one expects the spectral measures to be absolutely continuous. Indeed, the belief in such a simple and general description of the nature of the spectral measures dates back to the fundamental work of Aubry-André [1].[1] More recently, this conjecture shows up as Problem 6 of Simon's list of open problems in the theory of Schrödinger operators for the twenty-first century [26]. As we will discuss in more detail below, the result is known for α's satisfying a Diophantine condition. The strategy of the proof, however, clearly does not extend to the case of α's that are well approximated by rational numbers. Indeed, Simon points out in [26] that "one will need a new understanding of absolutely continuous spectrum to handle the case of Liouville α's."

A beautiful result of Kotani [19], which has not yet received the attention and exposure it deserves, shows that if the Lyapunov exponent vanishes in the spectrum, then absolute continuity of the IDS is equivalent to absolute continuity of the spectral measures for almost every θ; see also the survey [11] of Kotani theory and its applications. By [9], if the coupling is subcritical, the Lyapunov exponent is zero on the spectrum. We therefore obtain the following corollary.

Corollary 1. *If $|\lambda| < 1$, then the spectral measures of $H_{\lambda,\alpha,\theta}$ are absolutely continuous for almost every θ.*

This settles Problem 6 of [26], at least in an almost everywhere sense. [2]

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[1] In [1], it is actually proposed that spectral measures are absolutely continuous for $|\lambda| < 1$ and atomic for $|\lambda| > 1$ (with both regimes being linked in the heuristic reasoning). The problem turned out to be very subtle, and the claim for $|\lambda| > 1$ was soon shown to be wrong as stated (see further discussion below). This discovery did generate some doubts regarding the claim for $|\lambda| < 1$ (see Problem 5 in Section 11 of [24]) before optimism was regained with the work of Last [20].

[2] After this work was completed, an approach to proving absolutely continuous spectrum for every phase has been proposed by the first named author. The proposed solution does use (as an important step) the techniques developed here.

As mentioned before, the Main Theorem had been established already for certain ranges of parameters. Let us discuss in more detail which range of parameters could be covered by such methods, and which range of parameters will be treated by the new techniques introduced in this paper.

Due to the symmetries of the system, we may restrict our attention to $\lambda > 0$. It is known that $N_{\lambda,\alpha}(E) = N_{\lambda^{-1},\alpha}(\lambda^{-1}E)$ (Aubry duality). Thus in order to establish the Main Theorem it is enough to show that $N_{\lambda,\alpha}$ is absolutely continuous for $0 < \lambda < 1$.

In the "Diophantine case," the following argument has been successful. One establishes pure point spectrum for almost every phase for $H_{\lambda^{-1},\alpha,\theta}$, which implies by the strong version of Aubry duality [16] that the spectrum of $H_{\lambda,\alpha,\theta}$ is absolutely continuous for almost every phase, which obviously implies that $N_{\lambda,\alpha}$ is absolutely continuous.

Pure point spectrum for $H_{\lambda^{-1},\alpha,\theta}$ for almost every phase was indeed established by Jitomirskaya for all $\lambda^{-1} > 1$ under a full measure condition on α; see [17]. This result was recently strengthened by Avila and Jitomirskaya as follows. Let p_n/q_n be the continued fraction approximants to α and let

$$\beta = \beta(\alpha) = \limsup_{n\to\infty} \frac{\ln q_{n+1}}{q_n}. \tag{2}$$

If $\beta = 0$ and $\lambda^{-1} > 1$, then $H_{\lambda^{-1},\alpha,\theta}$ has pure point spectrum for almost every phase; see [2, Theorem 5.2]. As a consequence, the main theorem is known in the case $\beta = 0$.

However, it has been understood for a long time that this approach cannot work for all α [6]. Indeed, if $\beta > 0$ and $1 < \lambda^{-1} < e^{\beta}$, Gordon's Lemma [15] (and the formula for the Lyapunov exponent [9]) shows that there is no point spectrum at all! It is expected that pure point spectrum for almost every phase does hold when $\lambda^{-1} > e^{\beta}$. This is currently established when $\lambda^{-1} > e^{16\beta/9}$ [2]. In any event, the main theorem cannot be proven via the "localization plus duality" route when $\beta > 0$.

In this work, we provide a new approach to absolute continuity of the IDS via rational approximations. This approach works when $0 < \lambda < 1$ and $\beta > 0$. A feature of this approach is that absolutely continuous spectrum for almost every phase is obtained as a consequence of the absolute continuity of the IDS, rather than the other way around as is usually the case in the regime of zero Lyapunov exponent. In other words, our work establishes the first application of Kotani's gem (a sufficient condition for purely absolutely continuous spectrum in terms of the IDS and the Lyapunov exponent) from [19].

2. Preliminaries

2.1. $\mathrm{SL}(2,\mathbb{R})$**-action.** Recall the usual action of $\mathrm{SL}(2,\mathbb{C})$ on $\overline{\mathbb{C}}$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

In the following we restrict to matrices $A \in \mathrm{SL}(2,\mathbb{R})$.

Such matrices preserve $\mathbb{H} = \{z \in \mathbb{C}, \Im z > 0\}$. The Hilbert-Schmidt norm of

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is $\|A\|_{\mathrm{HS}} = (a^2+b^2+c^2+d^2)^{1/2}$. Let $\phi(z) = \frac{1+|z|^2}{2\Im z}$ for $z \in \mathbb{H}$. Then

$$\begin{aligned}
\phi(A \cdot i) &= \phi(\tfrac{ai+b}{ci+d}) \\
&= \frac{1+|\frac{ai+b}{ci+d}|^2}{2\Im \frac{ai+b}{ci+d}} \\
&= \frac{1+\frac{a^2+b^2}{c^2+d^2}}{2\Im \frac{(ai+b)(-ci+d)}{c^2+d^2}} \\
&= \frac{a^2+b^2+c^2+d^2}{2(ad-bc)}
\end{aligned}$$

and hence $\|A\|_{\mathrm{HS}}^2 = 2\phi(A \cdot i)$. Thus $\phi(z)$ is half the square of the Hilbert-Schmidt norm of an $\mathrm{SL}(2,\mathbb{R})$ matrix that takes i to z.

The rotation matrices

$$R_\theta = \begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix} \tag{3}$$

are characterized by the fact that they fix i. One easily checks that $\|R_\theta A\|_{\mathrm{HS}} = \|AR_\theta\|_{\mathrm{HS}} = \|A\|_{\mathrm{HS}}$. In particular $\phi(R_\theta z) = \phi(z)$.

We notice that $\phi(z) \ge 1$, $\phi(i) = 1$ and $|\ln \phi(z) - \ln \phi(w)| \le \mathrm{dist}_{\mathbb{H}}(z,w)$ where $\mathrm{dist}_{\mathbb{H}}$ is the hyperbolic metric on $\mathbb{H}$, normalized so that $\mathrm{dist}_{\mathbb{H}}(ai,i) = |\ln a|$ for $a > 0$.

If $|\mathrm{Tr} A| < 2$, then there exists a unique fixed point $z \in \mathbb{H}$, $A \cdot z = z$. Let $0 < \rho < 1/2$ be such that $\mathrm{Tr} A = 2\cos 2\pi\rho$. Let us show that

$$\phi(z) = \frac{1}{2\sin 2\pi\rho}(\|A\|_{\mathrm{HS}}^2 - 2\cos 4\pi\rho)^{1/2}, \tag{4}$$

so that

$$\phi(z) \le \frac{\sqrt{2}\|A\|_{\mathrm{HS}}}{2\sin 2\pi\rho}. \tag{5}$$

Let $B \in \mathrm{SL}(2,\mathbb{R})$ be such that $B \cdot i = z$. Then

$$A = BR_{\pm\rho}B^{-1} \tag{6}$$

since $B^{-1}AB$ fixes i and hence is a rotation that has the same trace as A. Write $B = RD$ with R a rotation and D diagonal,

$$D = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

(First stretch i suitably and then rotate it to z.) Then $DR_{\pm\rho}D^{-1}$ has a unique fixed point $R^{-1} \cdot z = D \cdot i$ and

$$\phi(z) = \phi(R^{-1} \cdot z) = (\lambda^2 + \lambda^{-2})/2 = \|D\|_{\mathrm{HS}}^2/2.$$

On the other hand,

$$\begin{aligned}\|A\|_{\mathrm{HS}}^2 &= \|DR_{\pm\rho}D^{-1}\|_{\mathrm{HS}}^2 \\ &= 2\cos^2 2\pi\rho + (\lambda^4+\lambda^{-4})\sin^2 2\pi\rho \\ &= 2\cos^2 2\pi\rho + (\|D\|_{\mathrm{HS}}^4 - 2)\sin^2 2\pi\rho \\ &= 2\cos 4\pi\rho + \|D\|_{\mathrm{HS}}^4 \sin^2 2\pi\rho \\ &= 2\cos 4\pi\rho + 4\phi(z)^2\sin^2 2\pi\rho,\end{aligned}$$

and we obtain (4).

2.2. **Lyapunov Exponent.** Fix $\lambda \in \mathbb{R}$, $\alpha \in \mathbb{R}$, $E \in \mathbb{R}$. Let

$$A(\theta) = A^{(\lambda,E)}(\theta) = \begin{pmatrix} E - 2\lambda\cos 2\pi\theta & -1 \\ 1 & 0 \end{pmatrix}. \tag{7}$$

Denote

$$A_n(\theta) = A_n^{(\lambda,\alpha,E)}(\theta) = A(\theta+(n-1)\alpha)\cdots A(\theta). \tag{8}$$

The Lyapunov exponent $L = L_{\lambda,\alpha}$ is defined as

$$L(E) = \lim_{n\to\infty} \frac{1}{n}\int_{\mathbb{R}/\mathbb{Z}} \ln\|A_n(\theta)\|\, d\theta. \tag{9}$$

By unique ergodicity of irrational rotations, for $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, we have $L(E) = \lim \frac{1}{n}\sup_\theta \ln\|A_n(\theta)\|$.

Theorem 1 ([9], Corollary 2)**.** *If $0 < \lambda < 1$, $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ and $E \in \Sigma_{\lambda,\alpha}$ then $L(E) = 0$.*

2.3. **The Almost Mathieu Operator.** We extend the definitions of the introduction to the case of rational frequencies p/q (here and in what follows, p/q will always denote the reduced fraction of a rational number). We can of course define $H_{\lambda,p/q,\theta}$ by the same formula as in the introduction; however, the several related quantities are no longer θ independent.

Let $N_{\lambda,p/q}$ be the average over θ of the integrated density of states of the operators $H_{\lambda,p/q,\theta}$. Then we still have the Thouless formula

$$L(E) = \int \ln|E'-E|dN(E'). \tag{10}$$

Let $\Sigma_{\lambda,p/q}$ be the union over θ of the spectra of $H_{\lambda,p/q,\theta}$ and let $\sigma_{\lambda,p/q}$ be the intersection over θ of the spectra of $H_{\lambda,p/q,\theta}$.

With those definitions, $\Sigma_{\lambda,\alpha}$ and $N_{\lambda,\alpha}$ are continuous in λ and α.

Theorem 2 ([5], Proposition 7.1)**.** *The Hausdorff distance between $\Sigma_{\lambda,\alpha}$ and $\Sigma_{\lambda,\alpha'}$ is at most $6(2\lambda)^{1/2}|\alpha-\alpha'|^{1/2}$, for $|\alpha-\alpha'| \le \frac{C}{\lambda}$, where $C > 0$ is some constant.*

2.4. **Periodic Case.** When $\alpha = p/q$, $\mathrm{Tr}A_q(\theta)$ is periodic of period $1/q$. Since $\mathrm{Tr}A_q(\theta) = -\lambda^q e^{2\pi i q\theta} - \lambda^q e^{-2\pi i q\theta} + \sum_{j=1-q}^{q-1} a_j e^{2\pi i j\theta}$ it follows that $a_j = 0$ for $0 < |j| < q$ and we have the Chambers formula

$$\mathrm{Tr}A_q(\theta) = -2\lambda^q\cos 2\pi q\theta + a_0 \tag{11}$$

where $a_0 = a_0(\lambda, p/q, E)$.

We have $\Sigma_{\lambda,p/q} = \{E : \inf_\theta |\mathrm{Tr}A_q(\theta)| \le 2\}$ and $\sigma_{\lambda,p/q} = \{E : \sup_\theta |\mathrm{Tr}A_q(\theta)| \le 2\}$. Bands of $\Sigma_{\lambda,p/q}$ are closures of the connected components of $\{E : \inf_\theta |\mathrm{Tr}A_q(\theta)| <$

$2\}$. Then there are q bands, $\Sigma_{\lambda,p/q}$ is the union of the bands, the bands can only touch, possibly, at the edges. Moreover, $\sigma_{\lambda,p/q}$ is non-empty if and only if $0<\lambda\le 1$, in which case it has q connected components and each band of $\Sigma_{\lambda,p/q}$ intersects in its interior a single connected component of $\sigma_{\lambda,p/q}$. (The facts above can be all deduced from Chambers formula, see [5] or [8].)

Theorem 3 ([5], Theorems 1 and 2)**.** *For $0<\lambda<1$, we have $|\sigma_{\lambda,p/q}| = 4-4\lambda$ and $|\Sigma_{\lambda,p/q}\setminus\sigma_{\lambda,p/q}| \le 4\pi\lambda^{q/2}$.*

Passage to the limit at irrational frequencies yields the lower bound of Thouless [27]

$$|\Sigma_{\lambda,\alpha}| \ge 4-4\lambda, \quad 0<\lambda<1,\ \alpha\in\mathbb{R}\setminus\mathbb{Q} \tag{12}$$

(of course, equality is now known to hold in the above formula, but we will not need it).

2.4.1. *Formulas for the IDS Inside a Band.* If $|\mathrm{Tr}A_q(\theta)|\le 2$, let $0\le\rho(\theta)\le 1/2$ be such that $\mathrm{Tr}A_q(\theta)=2\cos(2\pi\rho(\theta))$. Let also $\rho(\theta)=0$ if $\mathrm{Tr}A_q(\theta)>2$ and $\rho(\theta)=1/2$ if $\mathrm{Tr}A_q(\theta)<-2$. Let ρ be the average over θ of $\rho(\theta)$. Then if E belongs to the k-th band of $\Sigma_{\lambda,p/q}$, we have the formula

$$qN(E) = k-1+(-1)^{q+k-1}2\rho+\frac{1-(-1)^{q+k-1}}{2}. \tag{13}$$

This formula is immediate from the relation between the integrated density of states and the fibered rotation number; see [7] and [18].

If $|\mathrm{Tr}A_q(\theta)|<2$, let $m(\theta)$ be the fixed point of $A_q(\theta)$ in $\mathbb{H}$. Note that, by periodicity, we have $A(\theta)m(\theta)=m(\theta+p/q)$. Moreover,

$$\frac{d}{dE}N(E) = \frac{1}{2\pi}\int_{|\mathrm{Tr}A_q(\theta)|<2}\phi(m(\theta))\,d\theta. \tag{14}$$

This formula can be obtained for instance as a very simple case of the general formulas for absolutely continuous spectrum of [12].

2.5. **Derivative of the IDS in the Irrational Case.** The following result is a consequence of Theorem 1 and [12, 25].

Theorem 4. *Let $0<\lambda<1$, $\alpha\in\mathbb{R}\setminus\mathbb{Q}$. Then there exists a full Lebesgue measure subset $Y\subset\Sigma_{\lambda,\alpha}$ such that for every $E\in Y$, there exists a measurable function $\tilde m:\mathbb{R}/\mathbb{Z}\to\mathbb{H}$ such that $A(\theta)\cdot\tilde m(\theta)=\tilde m(\theta+\alpha)$ and*

$$\frac{d}{dE}N_{\lambda,\alpha}(E) = \frac{1}{2\pi}\int\phi(\tilde m(\theta))\,d\theta. \tag{15}$$

We refer the reader to [11, 19] for more information on the theory leading to the formulae (14) and (15); see especially [11, Theorem 5] and [19, Theorem 4.8].

3. Proof of the Main Theorem

As discussed in the introduction, it is enough to prove the following result. Recall the definition (2) of $\beta(\alpha)$.

Theorem 5. *Let $0<\lambda<1$, $\alpha\in\mathbb{R}\setminus\mathbb{Q}$. If $\beta(\alpha)>0$, then the IDS is absolutely continuous.*

The proof of this theorem will take up the remainder of this section. Throughout the proof, λ and α will be fixed.

Let $Y \subset \Sigma_{\lambda,\alpha}$ be as in Theorem 4. If $E \in Y$, let $\tilde{m}$ be as in Theorem 4. It is enough to prove that

$$\int_Y \frac{d}{dE} N_{\lambda,\alpha}\, dE = 1. \tag{16}$$

The hypothesis implies that

$$\left|\alpha - \frac{p}{q}\right| < e^{-(\beta - o(1))q} \tag{17}$$

for arbitrarily large q. Fix some p/q with this property and q large.

For a fixed energy E, write $A = A^{(\lambda,E)}$, $A_n = A_n^{(\lambda,p/q,E)}$ and $\tilde{A}_n = A_n^{(\lambda,\alpha,E)}$.

Let

$$c = \min\{\beta/2, -\ln \lambda/2\}.$$

Notice that $\max\{e^{-\beta q}, \lambda^q\} \leq e^{-2cq}$. Let

$$P_q = \{\rho \in [1/q, 1/2 - 1/q] : \exists a = a(\rho), b = b(\rho) \text{ positive integers with } a \text{ odd},$$
$$e^{cq/4} < b < e^{cq/2}, \text{ and } |4b\rho - a| < 10/b\}.$$

Define

$$X = X_{\lambda,p/q} = \{E \in \sigma_{\lambda,p/q} : \rho \in P_q\}.$$

Notice that

$$\begin{aligned}
|4b(\rho)\rho(\theta) - a(\rho)| &= |4b(\rho)\rho - a(\rho)| + |4b(\rho)||\rho(\theta) - \rho| \\
&\lesssim e^{-cq/4} + e^{cq/2}\lambda^q \\
&\lesssim e^{-cq/4} + e^{cq/2}e^{-2cq} \\
&= O(e^{-cq/4})
\end{aligned}$$

(where $|\rho - \rho(\theta)|$ is estimated using Chambers formula).

Lemma 3.1. *We have*

$$|N_{\lambda,p/q}(X_{\lambda,p/q})| = 1 - o(1). \tag{18}$$

Proof. By the Chambers formula (see §2.4) and (13),

$$|N_{\lambda,p/q}(\Sigma_{\lambda,p/q} \setminus \sigma_{\lambda,p/q})| = o(1),$$

so it suffices to show

$$|N_{\lambda,p/q}(\sigma_{\lambda,p/q} \setminus X_{\lambda,p/q})| = o(1).$$

This in turn follows once we show in each connected component B of $\sigma_{\lambda,p/q}$ that

$$|N_{\lambda,p/q}(B \setminus X_{\lambda,p/q})| = o(1/q).$$

Now,

$$|N_{\lambda,p/q}(B \setminus X_{\lambda,p/q})| \leq \frac{1 - 2|P_q|}{q}$$

by (13). Thus, it suffices to show that $|P_q| \to 1/2$.

This follows from the observation that numbers $\rho \in [0, 1/2]$ such that the denominators d_k of the best approximants of 4ρ satisfy $d_{k+1} < d_k^{4/3}$ for all k large enough have full Lebesgue measure on $[0, 1/2]$ and belong to $\liminf_{q\to\infty} P_q$. Thus, at least two denominators fall into the allowed window and one of them can be

used due to the fact that two successive numerators cannot both be even. If there were two consecutive even numerators, then by the recursion all earlier numerators must have been even; but the first one was 1 and hence odd. □

Lemma 3.2. *We have*

$$|\Sigma_{\lambda,p/q} \setminus \Sigma_{\lambda,\alpha}| \le e^{-(c-o(1))q}. \tag{19}$$

In particular,

$$|X_{\lambda,p/q} \setminus \Sigma_{\lambda,\alpha}| \le e^{-(c-o(1))q}. \tag{20}$$

Proof. The Lebesgue measure of $\Sigma_{\lambda,p/q}$ is $4-4\lambda+O(e^{-cq})$ by Theorem 3. Since $\Sigma_{\lambda,p/q}$ is the union of q intervals, Theorem 2 implies

$$|\Sigma_{\lambda,\alpha} \setminus \Sigma_{\lambda,p/q}| = O(qe^{-cq}). \tag{21}$$

Since $|\Sigma_{\lambda,\alpha}| \ge 4-4\lambda$ by (12),

$$\begin{aligned} |\Sigma_{\lambda,p/q} \setminus \Sigma_{\lambda,\alpha}| &= |\Sigma_{\lambda,p/q}| - |\Sigma_{\lambda,p/q} \cap \Sigma_{\lambda,\alpha}| \\ &= |\Sigma_{\lambda,p/q}| - (|\Sigma_{\lambda,\alpha}| - |\Sigma_{\lambda,\alpha} \setminus \Sigma_{\lambda,p/q}|) \\ &\le 4-4\lambda+O(e^{-cq}) - (4-4\lambda) + O(qe^{-cq}), \end{aligned}$$

and the result follows. □

If E belongs to the interior of $\sigma_{\lambda,p/q}$, let $m(\theta)$ be the fixed point of $A_q(\theta)$ in $\mathbb{H}$, as in §2.4.

Lemma 3.3. *We have*

$$\sup_{E\in X} \sup_{\theta} \ln \phi(m(\theta)) = o(q). \tag{22}$$

Proof. By Theorem 1, the Lyapunov exponent is zero in $\Sigma_{\lambda,\alpha}$. By unique ergodicity of rotations (see the comment before Theorem 1), this means that for every $E \in \Sigma_{\lambda,\alpha}$ and for every $\varepsilon > 0$, there exists $n_0(\varepsilon, E)$ such that

$$\ln \|A_n^{(\lambda,\alpha,E)}(\theta)\| < \varepsilon n$$

for every θ and every $E \in \Sigma_{\lambda,\alpha}$ for $n > n_0(\varepsilon,E)$. This obviously implies that there exists $\delta(\varepsilon,E) > 0$ such that if $|\alpha'-\alpha| < \delta(\varepsilon,E)$ and $|E'-E| < \delta(\varepsilon,E)$, then

$$\ln \|A_n^{(\lambda,\alpha',E')}(\theta)\| < \varepsilon n$$

for every θ and every $n_0(\varepsilon,E) < n \le 2n_0(\varepsilon,E)+1$, and hence, by subadditivity, for every $n > n_0(\varepsilon,E)$. By compactness of $\Sigma_{\lambda,\alpha}$, we conclude that there exists $\delta(\varepsilon) > 0$ and $n_0(\varepsilon) > 0$ such that if $|\alpha'-\alpha| < \delta(\varepsilon)$ and E is at distance at most $\delta(\varepsilon)$ of $\Sigma_{\lambda,\alpha}$, then

$$\ln \|A_n^{(\lambda,\alpha',E)}(\theta)\| < \varepsilon n$$

for every θ and $n > n_0(\varepsilon)$. If p/q is sufficiently close to α so that $q > n_0(\varepsilon)$ and $\Sigma_{\lambda,p/q}$ is contained in a $\delta(\varepsilon)$ neighborhood of $\Sigma_{\lambda,\alpha}$, it then follows that

$$\ln \|A_q^{(\lambda,p/q,E)}(\theta)\| < \varepsilon q$$

for every $E \in \Sigma_{\lambda,p/q}$, and in particular for every $E \in \sigma_{\lambda,p/q}$.

On the other hand, by definition of X, $|\mathrm{Tr} A_q| < 2 - 1/5q^2$ if $E \in X$. It now follows from (5) that $\ln \phi(m) = o(q)$. □

Lemma 3.4. *We have*

$$|N_{\lambda,p/q}(X \setminus Y)| = o(1). \tag{23}$$

Proof. Note that

$$|X \setminus Y| = |X \setminus \Sigma_{\lambda,\alpha}| \leq e^{-(c-o(1))q} \tag{24}$$

by (20). On the other hand,

$$\ln \frac{d}{dE} N_{\lambda,p/q}(E) = o(q). \tag{25}$$

over X by Lemma 3.3 and (14). Thus,

$$\frac{d}{dE} N_{\lambda,p/q}(E) = e^{o(q)}$$

and hence

$$|N_{\lambda,p/q}(X \setminus Y)| = e^{-(c-o(1))q} e^{o(q)} = e^{-(c-o(1))q},$$

from which the result follows. □

Lemma 3.5. *If $E \in X \cap Y$, then*

$$\ln \int \phi(\tilde{m}(\theta))\, d\theta > \ln \int \phi(m(\theta))\, d\theta - o(1). \tag{26}$$

We will give the proof of this lemma in the next section.

We can now easily conclude (16) and thus Theorem 5 and the Main Theorem. We have

$$\begin{aligned} \int_Y \frac{d}{dE} N_{\lambda,\alpha}(E)\, dE &\geq \frac{1}{2\pi} \int_{X \cap Y} \int \phi(\tilde{m}(\theta))\, d\theta\, dE \\ &\geq (1 - o(1)) \frac{1}{2\pi} \int_{X \cap Y} \int \phi(m(\theta))\, d\theta\, dE \\ &\geq (1 - o(1)) |N_{\lambda,p/q}(X \cap Y)| \\ &\geq 1 - o(1), \end{aligned}$$

where the first inequality is due to Theorem 4, the second is due to Lemma 3.5, the third is due to (14) and absolute continuity of the IDS in the periodic case, and the fourth is due to Lemmas 3.1 and 3.4.

4. Proof of Lemma 3.5

We keep the notation from the previous section.

Since $E \in X$, $\rho \in P_q$. Let a and b be as in the definition of P_q. Let us show that for every θ,

$$\frac{\phi(\tilde{m}(\theta)) + \phi(\tilde{m}(\theta + bq\alpha))}{2} > (1 - o(1)) \phi(m(\theta)), \tag{27}$$

which easily implies Lemma 3.5. The estimate (27) is obvious when $\phi(\tilde{m}(\theta)) > 2\phi(m(\theta))$, so we will assume from now on that $\phi(\tilde{m}(\theta)) \leq 2\phi(m(\theta)) \leq e^{o(q)}$.

Choose $B(\theta) \in \mathrm{SL}(2, \mathbb{R})$ with $B(\theta) \cdot i = m(\theta)$. Then, since $A(\theta) \cdot m(\theta) = m(\theta + p/q)$, it follows that

$$B(\theta + p/q)^{-1} A(\theta) B(\theta) \cdot i = i$$

and hence

$$A(\theta) = B(\theta + p/q) R_{\psi(\theta)} B(\theta)^{-1}. \tag{28}$$

By Lemma 3.3, $\ln \|B(\theta)\| = o(q)$ (recall that $\phi(m(\theta)) = \frac{1}{2}\|B(\theta)\|_{\mathrm{HS}}^2$ if $B(\theta)$ takes i to $m(\theta)$). We have

$$\prod_{i=q-1}^{0} R_{\psi(\theta+ip/q)} = B(\theta)A_q(\theta)B(\theta)^{-1} = R_{\varepsilon\rho(\theta)},$$

where ε is either 1 or -1. The first identity follows from (28) and the second from the definition of $\rho(\theta)$.

Lemma 4.1. *We have* $\|B(\theta)^{-1}\tilde{A}_{bq}(\theta)B(\theta) - R\| = O(e^{-cq/4})$ *where* $R = R_{\varepsilon/4}$ *or* $R = R_{-\varepsilon/4}$ *according to whether* $a = 1$ *or* $a = 3$ *modulo* 4.

Proof. Write

$$\tilde{A}_k(\theta) = \prod_{i=k-1}^{0} A(\theta + i\alpha) = \prod_{i=k-1}^{0} B(\theta + (i+1)p/q)Q_i B(\theta + ip/q)^{-1}. \tag{29}$$

That is,

$$\begin{aligned} Q_i &= B(\theta + (i+1)p/q)^{-1} A(\theta + i\alpha) B(\theta + ip/q) \\ &= B(\theta + (i+1)p/q)^{-1} A(\theta + ip/q) B(\theta + ip/q) + \\ &\quad + B(\theta + (i+1)p/q)^{-1}[A(\theta + i\alpha) - A(\theta + ip/q)]B(\theta + ip/q) \\ &= R_{\psi(\theta+ip/q)} + \\ &\quad + B(\theta + (i+1)p/q)^{-1}[A(\theta + i\alpha) - A(\theta + ip/q)]B(\theta + ip/q) \end{aligned}$$

Thus,

$$\|Q_i - R_{\psi(\theta+ip/q)}\| = e^{o(q)} \left(|i| e^{-(\beta-(o(1))q}\right) e^{o(q)}$$

and hence, for $0 \le i < bq$,

$$\|Q_i - R_{\psi(\theta+ip/q)}\| = O(e^{o(q)+\frac{c}{2}q-(\beta-(o(1))q+o(q)}) = O(e^{-(3c/2-o(1))q}).$$

Thus $\tilde{A}_{bq}(\theta) = B(\theta)QB(\theta)^{-1}$ where $Q = \prod_{i=bq-1}^{0} Q_i$ satisfies $\|Q - R_{\varepsilon b\rho(\theta)}\| = O(e^{-(c-o(1))q})$. Moreover, $|4b\rho(\theta) - a| \le O(e^{-cq/4})$, giving the result. $\square$

We will need an estimate in hyperbolic geometry, which was already used in a similar context in [4].

Lemma 4.2. *Let* z_1, z_2, z_3 *lie in the same hyperbolic geodesic of* $\mathbb{H}$, *with* z_3 *the midpoint between* z_1 *and* z_2. *Then* $\phi(z_1) + \phi(z_2) \ge 2\phi(z_3)$.

Proof. Let $2\ln k$ be the hyperbolic distance between z_1 and z_2. We may assume that z_1, z_2 and z_3 are obtained from ki, i/k and i by applying

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

Then,

$$\begin{aligned}
\phi(z_1)+\phi(z_2) &= \phi\left(A\begin{pmatrix} k^{1/2} & 0 \\ 0 & k^{-1/2}\end{pmatrix}\cdot i\right)+\phi\left(A\begin{pmatrix} k^{-1/2} & 0 \\ 0 & k^{1/2}\end{pmatrix}\cdot i\right)\\
&= \frac{1}{2}\left\|A\begin{pmatrix} k^{1/2} & 0 \\ 0 & k^{-1/2}\end{pmatrix}\right\|_{\mathrm{HS}}^2+\frac{1}{2}\left\|A\begin{pmatrix} k^{-1/2} & 0 \\ 0 & k^{1/2}\end{pmatrix}\right\|_{\mathrm{HS}}^2\\
&= \frac{1}{2}(ka^2+k^{-1}b^2+kc^2+k^{-1}d^2)+\frac{1}{2}(k^{-1}a^2+kb^2+k^{-1}c^2+kd^2)\\
&= \frac{1}{2}(k+k^{-1})(a^2+b^2+c^2+d^2)\\
&= \frac{1}{2}\frac{1+k^2}{k}\|A\|_{\mathrm{HS}}^2\\
&= \frac{1+k^2}{k}\phi(z_3),
\end{aligned}$$

and hence $\phi(z_1)+\phi(z_2)\geq 2\phi(z_3)$, as desired. □

We can now conclude. Since we have $\phi(m(\theta))=e^{o(q)}$, we obtain

$$|\ln\phi(\tilde{m}(\theta+bq\alpha))-\ln\phi(B(\theta)R_{1/4}B(\theta)^{-1}\cdot\tilde{m}(\theta))|=o(1), \tag{30}$$

by Lemma 4.1. More precisely, if $\tilde{B}(\theta)$ takes i to $\tilde{m}(\theta)$, then

$$\begin{aligned}
\phi(\tilde{m}(\theta+bq\alpha)) &= \phi(\tilde{A}_{bq}(\theta)\cdot\tilde{m}(\theta))\\
&= \phi(\tilde{A}_{bq}(\theta)\tilde{B}(\theta)\cdot i)\\
&= \frac{1}{2}\|\tilde{A}_{bq}(\theta)\tilde{B}(\theta)\|_{\mathrm{HS}}^2
\end{aligned}$$

and

$$\begin{aligned}
\phi(B(\theta)R_{1/4}B(\theta)^{-1}\cdot\tilde{m}(\theta)) &= \phi(B(\theta)R_{1/4}B(\theta)^{-1}\tilde{B}(\theta)\cdot i)\\
&= \frac{1}{2}\|B(\theta)R_{1/4}B(\theta)^{-1}\tilde{B}(\theta)\|_{\mathrm{HS}}^2
\end{aligned}$$

Notice that

$$\begin{aligned}
&\|\tilde{A}_{bq}(\theta)\tilde{B}(\theta)-B(\theta)R_{1/4}B(\theta)^{-1}\tilde{B}(\theta)\|_{\mathrm{HS}}=\\
&\qquad=\|B(\theta)[B(\theta)^{-1}\tilde{A}_{bq}(\theta)B(\theta)-R_{1/4}]B(\theta)^{-1}\tilde{B}(\theta)\|_{\mathrm{HS}}\\
&\qquad\leq\|B(\theta)\|_{\mathrm{HS}}\,\|B(\theta)^{-1}\tilde{A}_{bq}(\theta)B(\theta)-R_{1/4}\|_{\mathrm{HS}}\,\|B(\theta)^{-1}\tilde{B}(\theta)\|_{\mathrm{HS}}\\
&\qquad\leq e^{o(q)}e^{-cq/4}e^{o(q)}\\
&\qquad=e^{-cq/4+o(q)}.
\end{aligned}$$

The triangle inequality shows that

$$\|A-B\|\leq\varepsilon\Rightarrow\ln\frac{\|A\|}{\|B\|}\leq\ln(1+\varepsilon)$$

whenever $\|B\|\geq 1$.

It follows that

$$\begin{aligned}\ln \phi(\tilde{m}(\theta+bq\alpha)) - \ln \phi(B(\theta)R_{1/4}B(\theta)^{-1}\cdot \tilde{m}(\theta)) = \\ &= \ln \frac{1}{2}\|\tilde{A}_{bq}(\theta)\tilde{B}(\theta)\|^2_{\mathrm{HS}} - \ln \frac{1}{2}\|B(\theta)R_{1/4}B(\theta)^{-1}\tilde{B}(\theta)\|^2_{\mathrm{HS}} \\ &= 2\ln \frac{\|\tilde{A}_{bq}(\theta)\tilde{B}(\theta)\|_{\mathrm{HS}}}{\|B(\theta)R_{1/4}B(\theta)^{-1}\tilde{B}(\theta)\|_{\mathrm{HS}}} \\ &= 2\ln\left(1+e^{-cq/4+o(q)}\right) \\ &= o(1),\end{aligned}$$

which is (30).

Let us show that the points $z_1 = \tilde{m}(\theta)$, $z_2 = B(\theta)R_{1/4}B(\theta)^{-1}\cdot\tilde{m}(\theta)$ and $z_3 = m(\theta)$ are as in Lemma 4.2 (recall that $m(\theta) = B(\theta)\cdot i$). Since $B(\theta)$ preserves hyperbolic distance, it is enough to show this for the points

$$B(\theta)^{-1}\cdot\tilde{m}(\theta),\ R_{1/4}B(\theta)^{-1}\cdot\tilde{m}(\theta),\ i.$$

Map $\mathbb{H}$ to $\mathbb{D}$ and observe that i gets mapped to 0 and the other two points to diametrically opposite points. Thus, these points lie on a hyperbolic geodesic and 0 is the midpoint.

So by Lemma 4.2, we have

$$\frac{\phi(\tilde{m}(\theta)) + \phi(B(\theta)R_{1/4}B(\theta)^{-1}\tilde{m}(\theta))}{2} \geq \phi(m(\theta)). \tag{31}$$

Using (30) and (31) one gets (27). This completes the proof of Lemma 3.5.

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